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Estimates of Christoffel Functions of Generalized Freud-Type Weights*

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DEDICATED TO THE MEMORY OF GÉZA FREUD

Upper and lower bounds are found for the generalized Christoffel functions $\lambda_{n,p}(d\mu; x)$ $(0 < p < \infty)$ of Freud-type weights. These weights have the form

$$w_r(x) = |x|^r \exp(-Q(x)) \qquad (x \in \mathbb{R},\ r > -1)$$

with a singularity at the origin and non-compact support. The proof requires an inequality reducing weighted integrals of polynomials over $\mathbb{R}$ to integrals over compact intervals.

I. INTRODUCTION

Géza Freud initiated investigations into the polynomials orthogonal with respect to $W(x) = \exp\{-Q(x)\}$ with $Q(x)$ chosen as $x^{2k}/2k$ [2, 4–7]. Nevai [15, 17] and Sheen [19, 20] have successfully handled the cases $k=2$ and $k=3$, respectively, where, as in much of Freud's work, estimates of the Christoffel functions gave crucial information needed in bounding the orthogonal polynomials. Freud also used the bounds to find weighted Markov–Bernstein-type inequalities [3] when Q is a Freud exponent (see (2.1)). Recently Lubinsky [9], Mhaskar and Saff [14], and Zalik [22] have investigated similar weighted inequalities; further, Lubinski [10] and Mhaskar and Saff [13] have bounded the generalized Christoffel functions for a wider class of smooth weights. Both the bounds of the Christoffel functions and the weighted inequalities are used in Magnus' proof [11, 12] of the Freud conjecture [4].

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In this paper we will investigate the Christoffel functions of Freud-type weights that have a singularity at the origin, that is, weights of the form

$$w_r(x) = |x|^r \exp(-Q(x)) \qquad (-\infty < x < +\infty, r > -1),$$

with $Q(x)$ being a Freud exponent. We intend to use the estimates given below to find the asymptotics of orthogonal polynomials associated with these generalized Freud-type weights.

The organization of the paper is as follows: In Section II we define our notation; Section III contains the statements of the main results; Section IV is the proof of the integral inequality; Section V contains the derivation of the bounds; and, lastly, Section VI relates q_n (see (2.3)) to the largest zero and to the ratios of leading coefficients of the orthogonal polynomials associated with these weights.

II. Notation

The following notations will be observed throughout. $Q(x)$ will be called a "Freud exponent" when Q is an even function and satisfies:

$$\begin{aligned}
&\text{(i)} \quad Q'(t) > 0,\ Q''(t) \geqslant 0 \text{ for } t \in (0, \infty),\\
&\text{(ii)} \quad Q''(t) \text{ is continuous on } \mathbb{R},\\
&\text{(iii)} \quad Q'(2t)/Q'(t) > c_0 > 1 \text{ for } t \in (0, \infty),\\
&\text{(iv)} \quad tQ''(t)/Q'(t) \leqslant c \text{ for } t \in (0, \infty).
\end{aligned} \tag{2.1}$$

The weight function, $w_r(x)$, will then be $w_r(x) = |x|^r \exp\{-Q(x)\}$. The polynomials orthonormal with respect to w_r are $p_n(w_r; x) = \gamma_n x^n + \cdots$, denote the greatest zero of $p_n(x)$ by $x_{1n}(w_r)$ and let

$$a_n(w_r) = \gamma_{n-1}(w_r)/\gamma_n(w_r). \tag{2.2}$$

For n suitably large let q_n be defined by the equation

$$q_n Q'(q_n) = n \qquad (n > n_0). \tag{2.3}$$

By $\mathbb{P}_n$, denote the set of all polynomials with real coefficients of degree at most n. The generalized Christoffel functions of the distribution $d\mu$ are (see Nevai [16], where they were first introduced)

$$\lambda_{n,p}(d\mu; x) = \inf_{\pi \in \mathbb{P}_{n-1}} \left[\int_{\mathbb{R}} |\pi(t)|^p \, d\mu(t) / |\pi(x)|^p \right].$$

We note that, for the special case $p=2$, the following identity is well known (e.g., Freud [8, Theorem 1.4.1]:

$$\lambda_{n,2}(d\mu; x) = \left[\sum_{k=0}^{n-1} p_k^2(d\mu; x)\right]^{-1}.$$

Denote by $c_1, c_2, \ldots$ positive constants independent of x or n.

III. The Main Results

The first result is the main tool with which the bounds were obtained.

THEOREM 3.1. *Let $Q(x)$ be a Freud exponent and q_n be as defined in* (2.3), *then for a fixed $\theta > 0$,* and *p, r such that $\theta \leqslant p \leqslant \infty$ and $pr > -1$, there exist constants $\rho = \rho(\theta) \in (0, 1)$, $c = c(\theta, r)$, and $B > 0$ so that for all $n > n_0$,*

$$\|\pi(x)\, w_r(x)\|_{L_p(\mathbb{R})} \leqslant (1 + c\rho^n)^{1/\theta} \|\pi(x)\, w_r(x)\|_{L_p(-Bq_n, +Bq_n)}$$

where $\pi(x) \in \mathbb{P}_n$.

Remark. The above inequality can be significantly sharpened using the techniques of Potential Theory (e.g., see Mhaskar and Saff [14]). We have chosen the methods used for simplicity of exposition since they do produce results sharp enough for the purposes of the following theorems. We also note that using $q_n \sim q_{n_0}$ for $n < n_0$ and standard compactness arguments we can extend the inequality to $n = 1, 2, \ldots$.

With this "Infinite to Finite Range" inequality in hand we can proceed to the main results, upper and lower bounds of the generalized Christoffel functions; Nevai [18] was the first to use the method of reducing weights over the real line to compact intervals in order to estimate the Christoffel functions.

THEOREM 3.2. *Let $Q(x)$ be a Freud exponent with q_n as defined in* (2.3), *let $0 < p < \infty$ and $pr > -1$, then, for $w_r(x) = |x|^r \exp(-Q(x))$, for every ε, $0 < \varepsilon < 1$, there is a constant $A = A(\varepsilon)$, independent of x and n, such that*

$$w_r^{-p}(x)\, \lambda_{n,p}(w_r^p; x) \geqslant A(q_n/n)(1 + (q_n/n)/|x|)^{pr} \qquad (|x| \leqslant \varepsilon B q_n)$$

where B is the constant of Theorem 3.1.

THEOREM 3.3. *Let $Q(x)$ be a Freud exponent with q_n as defined in* (2.3), *let* $0<p<\infty$ *and* $pr>-1$, *then, for* $w_r(x)=|x|^r\exp(-Q(x))$, *there is a* $\partial>0$ *and constant* A', *independent of* x *and* n, *such that*

$$w_r^{-p}(x)\,\lambda_{n,p}(w_r^p;x)\leqslant A'(q_n/n)(1+(q_n/n)/|x|)^{pr} \qquad (|x|\leqslant\partial q_n).$$

We immediately obtain the following

COROLLARY 3.4. *Under the conditions of Theorems* 3.2 *and* 3.3

$$w_r^{-p}(x)\,\lambda_{n,p}(w_r^p;x)\sim(q_n/n)(1+(q_n/n)/|x|)^{pr} \qquad (|x|\leqslant\partial q_n).$$

Remark. We note that from the definition of Freud exponent, (ii) Q'' continuous is used for the lower bound but not for the upper bound while (iii) $Q'(2t)/Q'(t)>c_0$ is used for the upper bound and not the lower.

The relation of q_n to the polynomials $p_n(w_r;x)$ (see Freud [5]) is seen in

THEOREM 3.5. *Let* $Q(x)$ *be a Freud exponent with* q_n *as defined in* (2.3) *and let* $r>-1$; *define* $w_r(x)=|x|^r\exp\{-Q(x)\}$. *Let* $x_{1n}(w_r)$ *be the greatest zero of* $p_n(w_r;x)$ *and let* $a_n(w_r)$ *be defined by* (2.2). *Then we have*

$$x_{1n}(w_r)\sim q_n \qquad \text{and} \qquad a_n(w_r)\sim q_n.$$

IV. PROOF OF THE "INFINITE TO FINITE RANGE" INEQUALITY

Following the method of Lubinsky [10] we use Cartan's Lemma for the

Proof. (Theorem 3.1). If $\pi(x)\equiv 0$ the inequality is trivial. Let $\pi\in\mathbb{P}_n$, $n>n_0$, we can express

$$\pi(x)=c\prod_{i=1}^{m}(x-x_i); \qquad c\neq 0,\ 0\leqslant m\leqslant n;\ x_1,\ldots,x_m\in\mathbb{C}\ \text{with}\ |x_1|\leqslant\cdots\leqslant|x_m|.$$

Let q_n be defined by (2.3). Determine $j\geqslant 0$ such that for $1\leqslant i\leqslant j$, $|x_i|\leqslant 3q_{2n}/2$ and for $j<i\leqslant m$, $|x_i|>3q_{2n}/2$. If $|x|>Bq_{2n}$, $|u|\leqslant q_{2n}$, and $j<i\leqslant m$, then

$$|x-x_i|/|u-x_i|\leqslant(1+|x|/|x_i|)/(1-|u|/|x_i|)\leqslant 3(1+(2/3)(|x|/q_{2n})).$$

i.e.,

$$|x-x_i|/|u-x_i|\leqslant 5(|x|/q_{2n}). \tag{4.1}$$

If $|x|>Bq_{2n}$, $|u|\leqslant q_{2n}$, and $1<i\leqslant j$, then

$$|x-x_i|/|u-x_i|\leqslant(|x|+(3/2)\,q_{2n})/|u-x_i|\leqslant 2|x|/|u-x_i|. \tag{4.2}$$

Putting (4.1) and (4.2) together yields

$$|\pi(x)/\pi(u)| \leqslant \prod_{i=1}^{j} (2|x|/|u-x_i|) \prod_{i=j+1}^{m} (5|x|/q_{2n})$$
$$= 2^j 5^{m-j} (|x|^m/(q_{2n}^{m-j})) \left[\prod_{i=1}^{j} |u-x_i| \right]^{-1}.$$

We shall now apply Cartan's lemma (see, e.g., Baker [1, p. 174]) to $\{\prod_{i=1}^{j} |u-x_i|\}$ to obtain

$$|\pi(x)/\pi(u)| \leqslant 5^m [48|x|/q_{2n}]^m$$

for $|x| \geqslant Bq_{2n}$, $|u| \leqslant q_{2n}$, and $u \notin \mathscr{S} \subset \mathbb{R}$, where $\mathscr{S}$ is a set which can be covered by intervals, the sum of whose lengths is at most $q_{2n}/8$. Let $\mathscr{M} = (-q_{2n}, +q_{2n}) \backslash \mathscr{S}$, then $\mathscr{M}$ has Lebesgue measure at least $(15/8)\, q_{2n}$. So for $u \in \mathscr{M}$, $|x| \geqslant Bq_{2n}$,

$$|\pi(x)\, w_r(x)|/|\pi(u)\, w_r(u)| \leqslant 5^m [48|x|/q_{2n}]^m w_r(x)/w_r(u).$$

Let $c_1 = \min\{1, (3/8)^r\}$ and $u \in \mathscr{M}^* = \mathscr{M} \backslash (-(3/8)\, q_{2n}, +(3/8)\, q_{2n})$, then

$$|\pi(x)\, w_r(x)|/|\pi(u)\, w_r(u)| \leqslant 5^m [48|x|/q_{2n}]^m w_r(x)/[w_0(q_{2n})\, c_1 q_{2n}^r]$$
$$\leqslant [2^{8n}/c_1][q_{2n}/|x|]^{n-r} [|x|^{2n} w_0(x)/(q_{2n}^{2n} w_0(q_{2n}))].$$

But, by the maximality of $q_{2n}^{2n} w_0(q_{2n})$, we have

$$|\pi(x)\, w_r(x)|/|\pi(u)\, w_r(u)| \leqslant [2^{8n}/c_1][q_{2n}/|x|]^{n-r},$$

i.e., for $|x| \geqslant Bq_{2n}$, and $u \in \mathscr{M}^*$,

$$|\pi(x)\, w_r(x)| \leqslant [2^{8n}/c_1][q_{2n}/|x|]^{n-r} |\pi(u)\, w_r(u)|.$$

Therefore

$$|\pi(x)\, w_r(x)|^p \leqslant [2^{8n}/c_1]^p [q_{2n}/|x|]^{(n-r)p} \min_{u \in \mathscr{M}^*} |\pi(u)\, w_r(u)|^p,$$

or

$$|\pi(x)\, w_r(x)|^p \leqslant [2^{8n}/c_1]^p [q_{2n}/|x|]^{(n-r)p} (1/q_{2n}) \int_{\mathscr{M}^*} |\pi(u)\, w_r(u)|^p du$$
$$\leqslant [2^{8n}/c_1]^p [q_{2n}/|x|]^{(n-r)p} (1/q_{2n}) \int_{-q_{2n}}^{+q_{2n}} |\pi(u)\, w_r(u)|^p du.$$

Whence

$$\int_{|x|\geqslant Bq_{2n}} |\pi(x)\, w_r(x)|^p\, dx \leqslant 2^{8pn+1} B^{-(n-r)p+1} c_1^{-p} [p(n-r)-1]^{-1} \times \int_{-q_{2n}}^{+q_{2n}} |\pi(u)\, w_r(u)|^p du.$$

Thus for B suitably large and $n > n_0$

$$\int_{|x|\geqslant Bq_{2n}} |\pi(x)\, w_r(x)|^p dx \leqslant A[\rho_1^n/c_1]^p [pn]^{-1} \int_{-q_{2n}}^{-q_{2n}} |\pi(u)\, w_r(u)|^p du.$$

Now

$$\int_{\mathbb{R}} |\pi(x)\, w_r(x)|^p dx = \left[\int_{|x|\leqslant Bq_{2n}} + \int_{|x|\geqslant Bq_{2n}}\right] |\pi(x)\, w_r(x)|^p dx$$

thus

$$\int_{\mathbb{R}} |\pi(x)\, w_r(x)|^p dx \leqslant [1 + (c_1/(pn))\, \rho^n] \int_{|x|\leqslant Bq_{2n}} |\pi(x)\, w_r(x)|^p dx.$$

So we have

$$\|\pi(x)\, w_r(x)\|_{L_p(\mathbb{R})} \leqslant [1 + (c_1/(pn))\, \rho^n]^{1/p} \|\pi(x)\, w_r(x)\|_{L_p(-Bq_n, +Bq_n)}$$

choosing B possibly larger, since $q_{2n} < 2q_n$ (Freud [3, p. 22]). Fix $\theta > 0$ then for $0 < \theta \leqslant p < \infty$

$$\|\pi(x)\, w_r(x)\|_{L_p(\mathbb{R})} \leqslant [1 + (c_1/(\theta n))\, \rho^n]^{1/\theta} \|\pi(x)\, w_r(x)\|_{L_p(-Bq_n, +Bq_n)}.$$

By the continuity of $\|\cdot\|_{L_p}$ norms and the independence of the constants upon p, the limit as $p \to \infty$ may be taken and the inequality holds for $0 < \theta \leqslant p \leqslant \infty$. ▮

V. Proofs of the Upper and Lower Bounds of the Christoffel Functions

First, we shall require a technical lemma

LEMMA 5.1. *Let* $R_n(x) = \sum_{k=0}^{n-1} x^k/k!$ *then*

$$(3/4) \exp(x) \leqslant R_n(x) \leqslant (5/4) \exp(x) \qquad (|x| \leqslant n/5,\ n \geqslant 12).$$

Proof. From Taylor's theorem, we have, for $|x| \leqslant cn$,

$$|\exp(x) - R_n(x)| \leqslant (n!)^{-1} \max_{|x| \leqslant cn} \{\exp(x)|x|^n\} \leqslant (n!)^{-1} \exp(cn)(cn)^n.$$

Applying Stirling's approximation gives

$$|\exp(x) - R_n(x)| \leqslant \exp((c+1)n)\, c^n,$$

in particular, for $c = 1/5$,

$$|1 - \exp(-x)\, R_n(x)| \leqslant (8/9)^n. \quad \blacksquare$$

We shall now construct the polynomials that will be used to approximate $w_0(x)$ (as in Freud [3]).

LEMMA 5.2. *Let $Q(x)$ be a Freud exponent, q_n be defined by* (2.3), *and fix $x \in \mathbb{R}$. There exists a polynomial $S_n(x; t)$ such that*

(i) $S_n(t) \in \mathbb{P}_{2kn}(t)$ *for each fixed x and some integer $k = k(Q, B)$,*

(ii) $S_n(x; x) = w_0(x)$,

(iii) $0 < S_n(t) \leqslant (5/4)\, w_0(t)$ for $|t| \leqslant Bq_n$,

where B is the constant of Theorem 3.1.

Proof. Let $V_n(t) = Q'(x)(t-x) + [c_0 n/(2q_n^2)](t-x)^2$ for $t \in \mathbb{R}$. Define

$$S_n(t) = w_0(x)\, R_{kn}(-V_n(t)) \qquad (|t| \leqslant Bq_n),$$

then (i) and (ii) follow directly. Now to prove (iii): For $|t| \leqslant Bq_n$

$$\begin{aligned} |V_n(t)| &\leqslant |Q'(x)|\, 2Bq_n + [c_0 n/(2q_n^2)]\, 4B^2 q_n^2 \\ &\leqslant c_1 |Q'(q_n)|\, 2Bq_n + 2B^2 c_0 n \leqslant 2B[c_1 + Bc_0]\, n. \end{aligned}$$

Therefore, if k is a large enough positive integer, so that $k/5 \geqslant 2B[c_1 + c_0 B]$, then, by Lemma 5.1,

$$R_{kn}(-V_n(t)) \sim \exp(-V_n(t)) \qquad (|t| \leqslant Bq_n);$$

so that

$$S_n(t) = w_0(x)\, R_{kn}(-V_n(t)) \sim w_0(x) \exp(-V_n(t)),$$

and hence

$$S_n(t)\, w_0^{-1}(t) \sim \exp\{Q(t) - Q(x) - Q'(x)(t-x) - [c_0 n/2q_n^2)](t-x)^2\}.$$

Since Q'' is continuous, $Q(t)=Q(x)+Q'(x)(t-x)+Q''(\xi)(t-x)^2/2$ for some ξ between t and x, but, since Q is a Freud exponent, $|Q''(\xi)|\leqslant c_0 n/q_n^2$, and thus (iii) holds. ∎

We are now in a position to determine the lower bound.

Proof. (Theorem 3.2). Let $p>0$, fix r such that $pr>-1$, and let $n>12$. Then

$$\begin{aligned}\lambda_{n,p}(w_r^p;x) &= \inf_{\pi\in\mathbb{P}_{n-1}}\int_{\mathbb{R}}|\pi(t)|^p w_r^p(t)\,dt/[\pi(x)]^p\\ &\geqslant \inf_{\pi\in\mathbb{P}_{n-1}}\int_{+Bq_n}^{-Bq_n}|\pi(t)|^p w_r^p(t)\,dt/[\pi(x)]^p\\ &\geqslant c_1 w_0^p(x)\inf_{\pi\in\mathbb{P}_{n-1}}\int_{-Bq_n}^{+Bq_n}|\pi(t)\,S_{2kn}(t)|^p|t|^{pr}dt/[\pi(x)\,S_{2kn}(x)]^p\\ &\geqslant c_2 w_0^p(x)\,q_n^{pr+1}\inf_{R\in\mathbb{P}_{k'n-1}}\int_{-1}^{+1}|R(tBq_n)|^p|t|^{pr}dt/[R(x)]^p\\ &\geqslant c_2 w_0^p(x)\,q_n^{pr+1}\inf_{R^*\in\mathbb{P}_{k'n-1}}\int_{-1}^{+1}|R^*(t)|^p|t|^{pr}dt/[R^*(x/Bq_n])]^p,\end{aligned}$$

so that

$$\lambda_{n,p}(w_r^p;x)\geqslant c_2 w_0^p(x)\,q_n^{pr+1}\lambda_{k'n,p}(|t|^{pr}\chi_{[-1,+1]}(t)\,dt;x/Bq_n).$$

Using Nevai [16, Theorem 6.3.25] we have, for $|x|\leqslant\varepsilon Bq_n$ $(0<\varepsilon<1)$,

$$\lambda_{n,p}(w_r^p;x)\geqslant Aw_r^p(x)[q_n/n][1+B(q_n/n)(1/|x|)]^{pr}. \quad ∎$$

Now we shall construct the polynomials to estimate $w_0(x)$ for the upper bound.

LEMMA 5.3. *Let $x\in\mathbb{R}$ be fixed and let $n>12$. Then there exists a polynomial $S_n(x;t)$ and $\partial>0$ such that for $|x|\leqslant\partial q_n$ and $|t|\leqslant Bq_n$,*

(i) $S_n(t)\in\mathbb{P}_{[n/2]}(t)$,

(ii) $S_n(x;x)=w_0^{-1}(x)$,

(iii) $0<S_n(t)\,w_0(t)\leqslant 5/4$.

where B is the constant of Theorem 3.1 *and q_n is defined by* (2.3).

Proof. Define $S_n(x;t)=w_0^{-1}(x)\,R_m(Q'(x)(t-x))$ where $m=[n/2]$ and R_m is defined in Lemma 5.1, then (i) and (ii) follow immediately. For

$|x| \leqslant \partial q_n$ and $|t| \leqslant Bq_n$, we have $|t-x| \leqslant (B+\partial)\, q_n$; now, since Q' is increasing

$$|Q'(x)(t-x)| \leqslant Q'(\partial q_n)(B+\partial)\, q_n = [Q'(\partial q_n)/Q'(q_n)]\, q_n Q'(q_n)(B+\partial).$$

Since Q is a Freud exponent

$$[Q'(\partial q_n)/Q'(q_n)] \leqslant [Q'(q_n 2^{-k})/Q'(q_n)] \leqslant c_0^{-k}.$$

Thus we can take $\partial > 0$ so small that

$$|Q'(x)(t-x)| \leqslant c_0^{-k} n(B+\partial) \leqslant n/20 \leqslant m/5,$$

therefore, by Lemma 5.1 and the convexity of Q,

$$S_n(t) \leqslant c w_0^{-1}(x) \exp\{(Q'(x)(t-x)\} \leqslant c \exp\{Q(t)\} = c w_0^{-1}(t). \quad ∎$$

Let us proceed to the

Proof (Theorem 3.3). As before let $p > 0$, fix r such that $pr > -1$, and let $n > 12$. Then

$$\lambda_{n,p}(w_r^p; x) = \inf_{\pi \in \mathbb{P}_{n-1}} \int_{\mathbb{R}} |\pi(t)|^p w_r^p(t)\, dt/[\pi(x)]^p$$

$$\leqslant c_1 \inf_{\pi \in \mathbb{P}_{n-1}} \int_{-Bq_n}^{+Bq_n} |\pi(t)|^p w_r^p(t)\, dt/[\pi(x)]^p,$$

which, applying Lemma 5.3, is

$$c_1 \inf_{R \in \mathbb{P}_{[n/2]}} \int_{-Bq_n}^{+Bq_n} |R(t)\, S_n(t)\, w_0(t)|^p |t|^{pr} dt/[R(x)\, S_n(x)]^p$$

$$\leqslant c_2 w_0^p(x) \inf_{R \in \mathbb{P}_{[n/2]}} \int_{-Bq_n}^{+Bq_n} |R(t)|^p |t|^{pr} dt/[R(x)]^p.$$

We apply the same change of variables as in the derivation of the lower bound to obtain

$$\leqslant c_3 w_0^p(x)\, q_n^{pr+1} \inf_{R^* \in \mathbb{P}_{[n/2]}} \int_{-1}^{+1} |R^*(u)|^p |u|^{pr} du/[R^*(x/Bq_n)]^p$$

so that

$$\lambda_{n,p}(w_r^p; x) \leqslant c_3 w_0^p(x)\, q_n^{pr+1} \lambda_{[n/2],p}(|t|^{pr} \chi_{[-1,+1]}(t)\, dt; x/Bq_n).$$

Once more using Nevai [16, Theorem 6.3.25] we have, for $|x| \leqslant \partial Bq_n$,

$$\lambda_{n,p}(w_r^p; x) \leqslant A' w_r^p(x) [q_n/n] [1 + B(q_n/n)(1/|x|)]^{pr}. \quad ∎$$

VI. Connections to the Orthonormal Polynomials $p_n(w_r; x)$

While Freud originally used the property that $q_{2n}^{2n} Q'(q_{2n})$ maximized $xQ'(x)$, there are other significant relations concerning q_n.

Lemma 6.1. *Let $x_{1n}(w_r)$ denote the greatest zero of the orthonormal polynomial, $p_n(w_r; x)$, and q_n be defined by* (2.3); *then*

$$\limsup_{n \to \infty} x_{1n}(w_r)/q_n \leqslant \text{const.}$$

Proof. From a well-known result of Chebyshev (see, e.g., Szegö [21, p. 187]) we have

$$x_{1n}(w_r) = \max_{\pi \in \mathbb{P}_n} \left[\int_{\mathbb{R}} x\pi^2(x)\, w_r(x)\, dx \Big/ \int_{\mathbb{R}} \pi^2(x)\, w_r(x)\, dx \right].$$

According to Theorem 3.1

$$\int_{\mathbb{R}} |x| \pi^2(x)\, w_r(x)\, dx \leqslant [1 + c\rho^{2n+1}] \int_{-Bq_n}^{+Bq_n} |x|\, \pi^2(x)\, w_r(x)\, dx$$

or

$$\leqslant 2[1 + c\rho^{2n+1}]\, Bq_n \int_{-Bq_n}^{+Bq_n} \pi^2(x)\, w_r(x)\, dx,$$

and the result is seen to hold. ∎

Lemma 6.2. *Let $r > -1$. Then*

$$\gamma_n(w_r)/\gamma_{n-1}(w_r) = (n + r\Delta_n)^{-1} \int_{-\infty}^{\infty} p_n(x)\, p_{n-1}(x)\, Q'(x)\, w_r(x)\, dx;$$

$$\Delta_n = \sin^2(n\pi/2).$$

Remark. For $Q(x) = |x|^\beta$ Lemma 6.2 was proven for $r \geqslant 0$ and $\beta > 0$ in Freud [6] and for $r > -1$ and $\beta \geqslant 1$ in Nevai [18].

Proof. First integrate directly

$$\int_{-\infty}^{\infty} p_n(x)\, p_{n-1}(x)\, w_r(x)\, dx = \int_{-\infty}^{\infty} (n\gamma_n x^{n-1} + \cdots)\, p_{n-1}(x)\, w_r(x)\, dx$$

$$= \int_{-\infty}^{\infty} (n(\gamma_n/\gamma_{n-1})\, p_{n-1}(x) + \pi_{n-2}(x))\, p_{n-1}(x)\, w_r(x)\, dx = n(\gamma_n/\gamma_{n-1}) \tag{6.1}$$

where $\pi_{n-2}(x) \in \mathbb{P}_{n-2}$, the last equality holding by virtue of orthogonality.

Now integrate by parts

$$\int_{-\infty}^{\infty} p_n'(x)\, p_{n-1}(x)\, w_r(x)\, dx = -\int_{-\infty}^{\infty} p_n(x)(p_{n-1}(x)\, w_r(x))' dx$$
$$= \int_{-\infty}^{\infty} p_n(x)\, p_{n-1}(x)\, Q'(x)\, w_r(x)\, dx$$
$$-r \int_{-\infty}^{\infty} p_n(x)\, p_{n-1}(x)\, x^{-1} w_r(x)\, dx. \qquad (6.2)$$

Since $w_r(x)$ is an even weight, p_n is an even/odd polynomial as n is even/odd, resp., therefore

$$\int_{-\infty}^{\infty} p_n(x)\, p_{n-1}(x)\, x^{-1} w_r(x)\, dx = (\gamma_n/\gamma_{n-1})\, \Delta_n.$$

Combining (6.1) and (6.2), the result follows. ∎

LEMMA 6.3. *Let $r > -1$, $n > n_0$, and $a_n(w_r) = \gamma_{n-1}(w_r)/\gamma_n(w_r)$, then*

$$Aq_n \leqslant a_n(w_r),$$

where A is an absolute constant.

Proof. From Lemma 6.2 we have

$$\gamma_n(w_r)/\gamma_{n-1}(w_r) = (n + r\Delta_n)^{-1} \int_{-\infty}^{\infty} p_n(x)\, p_{n-1}(x)\, Q'(x)\, w_r(x)\, dx.$$

Since Q is a Freud exponent, for $x > 0$

$$Q'(x) = Q'(q_n) \exp\{\log(Q'(x)) - \log(Q'(q_n))\}$$
$$Q'(q_n) \exp\left\{\int_{q_n}^{x} (Q''(t)/Q'(t))\, dt\right\}$$
$$\leqslant Q'(q_n) \exp\left\{\int_{q_n}^{x} (c/t)\, dt\right\} = Q'(q_n) |x/q_n|^c$$

with c being the constant of (2.1)(iv), whereupon

$$\gamma_n(w_r)/\gamma_{n-1}(w_r) \leqslant c_1 (n + r\Delta_n)^{-1} Q'(q_n) \int_{-\infty}^{+\infty} |p_n(x)\, p_{n-1}(x)|\, |x/q_n|^c w_r(x)\, dx.$$

We now apply Theorem 3.1 to obtain

$$\leqslant c_2(n+r\Delta_n)^{-1}Q'(q_n)\int_{-Bq_n}^{+Bq_n}|p_n(x)\,p_{n-1}(x)|\,|x/q_n|^c w_r(x)\,dx,$$

so that

$$\gamma_n(w_r)/\gamma_{n-1}(w_r)\leqslant c_3 n^{-1}Q'(q_n)\int_{-Bq_n}^{+Bq_n}|p_n(x)\,p_{n-1}(x)|\,w_r(x)\,dx,$$

i.e.,

$$\leqslant c_3 n^{-1}Q'(q_n)=c_3/q_n.$$

The last equality follows from the definition of q_n. ∎

Proof (Theorem 3.5). The inequality

$$\text{const}\ q_{n-1}\leqslant a_{n-1}\leqslant \max_{1\leqslant j\leqslant n-1} a_j\leqslant x_{1n}\leqslant 2\max_{1\leqslant j\leqslant n-1} a_j\leqslant 2x_{1n}\leqslant \text{const}\ q_n$$

follows from (Freud [6, Theorem 1])

$$\max_{1\leqslant j\leqslant n-1} a_j\leqslant x_{1n}\leqslant 2\max_{1\leqslant j\leqslant n-1} a_j$$

and Lemmas 6.1 and 6.3. Since q_n is an increasing function of n and $1<q_{2n}/q_n<2$ (Freud [3, p. 22]) the Theorem holds.

Remark. When Q is an even polynomial of degree $2m$ then $q_n \sim n^{1/2m}$ and given that $A_n(w_r)=a_n(w_r)/(n^{1/2m})$ has a limit, it is an easy calculation to find the value. Following the method of Freud [4] we integrate $\int p_n'(x)\,p_{n-1}(x)\,w_r(x)\,dx$ in two ways (as in Lemma 6.2) and we arrive at the recurrence relation for $a_n(w_r)$

$$n+r\sin^2(n\pi/2)=2a_n\sum_{k=1}^{m} kd_{2k}\int_{-\infty}^{+\infty} x^{2k-1}p_n(x)\,p_{n-1}(x)\,w_r(x)\,dx$$

where $Q(x)=\sum d_{2k}x^{2k}$; now, noting that the "order" of each of the integrals is $\sim C_{2k-1,k}a_n^{2k-1}$ ($C_{i,j}$ being the binominal coefficient), we find

$$\lim_{n\to\infty} a_n(w_r)/(n^{1/2m})=(2m\,d_{2m}C_{2m-1,m})^{-1/2m}$$

which is consistent with the Freud conjecture [4] (recently proven by A. Magnus [11] for the case $Q(x)=x^{2m}$, also see Magnus [12], where Freud's conjecture was discussed for $Q(x)=|x|^r$, $r>1$).

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